Total No. of Printed Pages—7

3 SEM TDC MTMH (CBCS) C 6

2025

(Nov/Dec)

MATHEMATICS

(Core)

Paper : C–6

(Group Theory—I)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. (a) Find the inverse of the element

$$\begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$$

in $GL(2, \mathbb{Z}_{11})$. 1

(b) State True or False : 1

The set {0, 1, 2, 3} is not a group under multiplication modulo 4.

(c) Prove that in a group G, there is only one identity element. 3

(d) Prove that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}\ \forall\ a, b \in G$. 4

(e) Describe the dihedral group D_4. Show that it is a group together with the operation composition. Is it an Abelian group? Justify. 6

Or

Describe the group of symmetries of a rectangle. Show that it is a group together with the operation composition. Is it Abelian? Justify.

2. (a) Give an example of a non-Abelian group having finite order. 1

(b) Find the order of all the elements of $U(10)$. 2

(c) Let G be an Abelian group with identity e. Show that $H = \{x \in G \mid x^2 = e\}$ is a subgroup of G. 3

(d) Define centre, $z(G)$ of a group G. Show that it is a subgroup of G. 1+3=4

(e) Let H be a non-empty finite subset of a group G and H is closed under the operation of G. Show that H is a subgroup of G. 5

Or

Let

$$G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \ a, b, c, d \in \mathbb{Z} \right\}$$

under addition. Let

$$H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G \ \middle| \ a+b+c+d = 0 \right\}$$

Prove that H is a subgroup of G. What if 0 is replaced by 1?

3. *(a)* State True or False : 1

$U(8)$ is a cyclic group.

(b) Show that $\mathbb{Z}_{10} = \langle 3 \rangle$. 2

(c) Find all the generators of $\mathbb{Z}_6$. 2

(d) Let

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix} \text{ and}$$

$$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{bmatrix}$$

Compute α^{-1}, $\alpha\beta$ and $\beta\alpha$. 3

(e) Let G be a finite group and let $a \in G$. Then prove that $a^{|G|} = e$. 3

(f) Let G be a group and H be a subgroup of G. Let $a \in G$. Prove that $aH = H$ if and only if $a \in H$. 4

(g) State and prove Lagrange's theorem for finite group. 1+4=5

Or

Show that every permutation of a finite set can be written as a cycle or as a product of disjoint cycles. 5

4. *(a)* Define normal subgroup. 1

(b) Find the order of each element in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. 2

(c) Prove that $SL(2, \mathbb{R})$ is a normal subgroup of $GL(2, \mathbb{R})$. 3

(d) Determine the number of elements of order 5 in $\mathbb{Z}_{25} \oplus \mathbb{Z}_5$. 4

(e) Let G and H be finite cyclic groups. Then prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime. 5

Or

Let G be a group and let $z(G)$ be the centre of G. Prove that if $G|z(G)$ is cyclic, then G is Abelian.

5. (a) Define kernel of a homomorphism. 1

(b) State True or False : 1

The kernel of an isomorphism is the identity.

(c) Let ϕ be a homomorphism from a group G to a group $\overline{G}$ and let H be a subgroup of G. Then prove the following : 2+2=4

(i) $\phi(H)$ is a subgroup of $\overline{G}$

(ii) If H is Abelian, then $\phi(H)$ is Abelian

(d) Let ϕ be a group homomorphism from a group G to a group $\overline{G}$. Then show that $\ker\phi$ is a normal subgroup of G. 4

(e) Let ϕ be a group homomorphism from G to $\overline{G}$. Then prove that $G/\ker\phi \approx \phi(G)$. 5

Or

Let G be a group of permutations. For each σ in G, define

$$\operatorname{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation} \end{cases}$$

Prove that sgn is a homomorphism from G to the multiplicative group $\{+1, -1\}$. What is the kernel? 1+4=5

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