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3 SEM TDC MTMH (CBCS) C 6

2023

(Nov/Dec)

MATHEMATICS

(Core)

Paper : C–6

(Group Theory—I)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* In $GL(2, Z_{11})$, find det $\begin{bmatrix} 9 & 4 \\ 2 & 6 \end{bmatrix}$. 1

(b) Show that {1, 2, 3}, under multiplication modulo 4 is not a group. 1

(c) Show that the inverse of every element in a group is unique. 2

(d) Write out a complete Cayley table for D_3. Is D_3 Abelian? 2+1=3

(e) Let G be a group and $a \in G$. Then prove that $O(a) = O(x^{-1}ax), \forall x \in G$. 3

(f) Show that the set of six transformations $f_1, f_2, f_3, f_4, f_5, f_6$ on the set of complex numbers defined by

$$f_1(z) = z, \quad f_2(z) = \frac{1}{z}, \quad f_3(z) = 1 - z$$

$$f_4(z) = \frac{z}{z-1}, \quad f_5(z) = \frac{1}{1-z}, \quad f_6(z) = \frac{z-1}{z}$$

form a finite non-Abelian group under composite of functions. 5

2. (a) In the group Z, find $\langle 12, 18, 45 \rangle$. 1

(b) Let G be a group and let a be any element of G. Then prove that $\langle a \rangle$ is a subgroup of G. 2

(c) Define product of two subgroups of a group and write the condition that the product of two subgroups will be a subgroup of that group. 1+1=2

(d) In the group Z_{12}, find $|a|$ and $|a+b|$ if $a = 6$ and $b = 2$. 2

(e) Define normalizer of an element of a group and also prove that the normalizer $N(a)$ of $a \in G$ is a subgroup of G. 1+2=3

(f) Prove that a non-empty subset H of a finite group G is a subgroup of G if and only if $HH = H$. 5

Or

Prove that the union of two subgroups of a group is a subgroup of the group if and only if one is contained in the other.

3. (a) Give an example of a cyclic group whose order is not prime. 1

(b) State Fermat's little theorem. 1

(c) Express the following permutation as a product of disjoint cycles. Also find whether it is even or odd : 1+1=2

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 7 & 3 & 1 & 4 & 2 & 6 \end{pmatrix}$$

(d) Prove that an infinite cyclic group has exactly two generators. 3

(e) If a permutation α can be expressed as a product of an even (odd) number of transpositions, then prove that every decomposition of α into a product of transpositions must have even (odd) number of transpositions. 3

(f) Prove that the set A_n of all even permutations on a set S having $n \geq 2$ elements is a subgroup of S_n of order $\frac{n!}{2}$. 5

(g) Let G be a group and H be a subgroup of G. Let $a,\ b \in G$. Then prove that

(i) $Ha = Hb$ if and only if $ab^{-1} \in H$

(ii) Ha is a subgroup of G iff $a \in H$ 5

Or

Let G be a group and H, K be two subgroups of G. Then prove that

$$O(HK) = \frac{O(H)O(K)}{O(H \cap K)}$$

4. *(a)* What is the order of the element $(g_1, g_2, \cdots, g_n)$ of the external direct product of the groups $(G_1, G_2, \cdots, G_n)$? 1

(b) Prove that the quotient group of an Abelian group is Abelian. 2

(c) Prove that $Z_2 \oplus Z_3$ is cyclic. 2

(d) If H is a subgroup of a group G such that $i_G(H) = 2$, then prove that H is normal subgroup in G. 5

(e) Let G be a group and $Z(G)$ be the centre of G. If $\frac{G}{Z(G)}$ is cyclic, then prove that G is Abelian. 5

Or

Let G and H be finite cyclic groups. Then prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime.

5. (a) Let f be an isomorphic mapping of a group G into a group G'. Then prove that the f-image of the inverse of an element a of G is the inverse of the f-image of a. 2

(b) Let G be any group and a be any fixed element in G. Define a mapping $f : G \to G$ by $f(x) = axa^{-1}, \forall\, x \in G$. Then prove that f is an isomorphism of G onto itself. 3

(c) Prove that the relation of isomorphism in the set of all groups is an equivalence relation. 5

(d) Show that the multiplicative group $G = \{1, \omega, \omega^2\}$ is isomorphic to the permutation group $G' = \{I, (abc), (acb)\}$ on three symbols a, b, c. 5

Or

If H and K be two normal subgroups of G such that $H \subseteq K$, then prove that

$$\frac{G}{K} \cong \frac{G/H}{K/H}$$

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