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5 SEM TDC DSE MTH (CBCS)
2.1/2.2/2.3/2.4 (H)

2023

(November)

MATHEMATICS

(Discipline Specific Elective)

(For Honours)

Paper : DSE–2.1/2.2/2.3/2.4

The figures in the margin indicate full marks for the questions

Paper : DSE–2.1

(Mathematical Modelling)

Full Marks : 60
Pass Marks : 24

Time : 3 hours

1. (a) What do you mean by an ordinary point of the following differential equation? 1

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0$$

(b) Define Bessel's equation of order zero. 1

(Turn Over)

2. (a) Show that $x=0$ is a regular singular point of the differential equation

$$2x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+(x-5)y=0$$ 3

(b) Find the power series solution near $x=0$ of the differential equation

$$\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2+2)y=0$$

in powers of x. 6

Or

Solve the following Bessel's equation :

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0$$

3. (a) If $L\{F(t)\}=f(s)$, then prove that

$$L\{F(at)\}=\frac{1}{a}f\left(\frac{s}{a}\right)$$ 2

(b) Prove that

$$L\{-a\sin at\}=-\frac{a^2}{s^2+a^2}$$ 3

(Continued)

(c) Evaluate the following using convolution theorem (any *one*) : 4

(i) $L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$

(ii) $L^{-1}\left\{\frac{1}{(s-2)(s^2+1)}\right\}$

(d) Solve the initial value problem using Laplace transform, $y''+y=t\cos t$ with $y(0)=0,\ y'(0)=0$. 5

4. (a) Write two characteristics of Monte Carlo simulation technique. 2

(b) Write the algorithm that gives the sequence of calculations needed for a general computer simulation of Monte Carlo technique for finding the area under a curve. 3

5. (a) Describe the middle-square method for generating random numbers. Write two disadvantages of middle square method. 3+2=5

(b) Use linear congruence method to generate a sequence of 10 random numbers with $x_0=27$, $a=17$, $b=43$ and $m=100$ by the rule 5

$$x_{n+1}=(ax_n+b)\bmod(m)$$

(Turn Over)

6. Write a short note on any *one* of the following : 5

(a) Morning rush hour queuing model

(b) Harbor model with example

7. Answer any *one* of the following : 6

(a) A firm makes two products X and Y, and has a total production capacity of 9 tonnes per day. Both X and Y require the same production capacity. The firm has a permanent contract to supply at least 2 tonnes of X and at least 3 tonnes of Y per day to another company. Each tonne of X requires 20 machine hours of production time and each Y requires 50 machine hours of production time. The daily maximum possible number of machine hours is 360. All of the firm's output can be sold. The profit made is ₹ 80 per tonne of X and ₹ 120 per tonne of Y. Solve the problem by using graphical method to determine the production schedule that yields the maximum profit.

(Continued)

(b) Using simplex method to solve the following linear programming model :

$$\text{Maximize } Z = 4x_1 + 3x_2$$

subject to

$$2x_1 + x_2 \le 1000$$
$$x_1 + x_2 \le 800$$
$$x_1 \le 400, \; x_2 \le 700$$

and $x_1, \; x_2 \ge 0$

8. A company wants to produce three products—A, B and C. The per unit profit on these products is ₹ 4, ₹ 6 and ₹ 2 respectively. These products require two types of resources—manpower and raw material. The LP model formulated for determining the optimal product is as follows :

$$\text{Maximize } Z = 4x_1 + 6x_2 + 2x_3$$

subject to the constraints

(i) $x_1 + x_2 + x_3 \le 3$ (Manpower constraint)

(ii) $x_1 + 4x_2 + 7x_3 \le 9$ (Raw material constraint)

where x_1, x_2, x_3 are the numbers of units of products A, B, C respectively to be produced, and $x_1, x_2, x_3 \ge 0$.

(Turn Over)

(a) Find the optimal product mix and the corresponding profit of the company. 4

(b) Find the range of the profit contribution of product A in the objective function such that current optimal product mix remains unchanged. 5

Paper : DSE–2.2

(Mechanics)

Full Marks : 80
Pass Marks : 32

Time : 3 hours

1. (a) Determine the moment about the origin O of the force $\vec{F} = -5N\hat{i} - 2N\hat{j} + 3N\hat{k}$ which acts at a point A. The position vectors of A are (i) $\vec{r} = 4m\hat{i} - 2m\hat{j} - 1m\hat{k}$ and (ii) $\vec{r} = -8m\hat{i} + 3m\hat{j} + 4m\hat{k}$. 2+2=4

(b) Forces $\vec{P}$, $\vec{Q}$, $\vec{R}$ acting along $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, where O is the circumcentre of the triangle ABC, are in equilibrium. Show that

$$\frac{P}{a^2(b^2+c^2-a^2)} = \frac{Q}{b^2(c^2+a^2-b^2)} = \frac{R}{c^2(a^2+b^2-c^2)}$$

6

(Continued)

Or

Forces $\vec{P}$, $\vec{Q}$, $\vec{R}$ acting along $\vec{IA}$, $\vec{IB}$, $\vec{IC}$, where I is the incentre of the triangle ABC, are in equilibrium. Show that

$$P:Q:R = \cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}$$

(c) Show that two couples in the same plane whose moments are equal and of the same sign are equivalent to one another. 6

(d) An electric light fixture weighing 15 N hangs from a point C, by two strings AC and BC. AC is inclined at 60° to the horizontal and BC at 45° to the vertical. Draw the free body diagram and determine the forces in the strings AC and BC. 4

2. (a) Write down the Coulomb's laws of friction. 2

(b) An automobile is on a roadway inclined at an angle θ with the horizontals. If the coefficient of static and dynamic frictions between the tyres and road are 0·6 and 0·5 respectively, find the maximum inclination 'θ_{max}' at which car can climb at uniform speed. It has a rear-wheel drive and a total loaded weight of 3600 kg. 6

(Turn Over)

(c) Find I_{xx}, I_{yy} and I_{xy} for the area bounded by $y = e^x$ and $y = -e^x$. 2+2+2=6

Or

Find the centroid of the region bounded by $y^2 = 2x$, the line $\frac{x}{10} + \frac{y}{7} = 1$ and y-axis. 6

(d) State and prove the theorem of Pappus-Guldinus. 2+4=6

(e) Establish the relation between second moments and product of inertia. 5

3. (a) What do you mean by conservative force field? Show that in a conservative force field, $\vec{F} = -\nabla V$, where the symbols have their usual meanings. 2+3=5

(b) Show that the kinetic energy of a system for some reference is equal to the sum of kinetic energy of the total mass moving relative to that reference with the velocity of the mass centre and kinetic energy of the motion of the particles relative to the mass centre. 7

(Continued)

Or

Show that the moment of the resultant force on a particle about a point, fixed in an inertial reference, equal to the time rate of change of moment of the linear momentum of the particle relative to the inertial reference frame.

(c) Derive the moment of momentum equation for a system of particle. 6

(d) Establish the relationship between time derivatives of a vector for different references moving arbitrarily relative to each other. 6

(e) Show that the kinetic energy of a system of particles is equal to the sum of the kinetic energy of the mass centre and the kinetic energy of the system in its motion relative to moving frame of reference. 5

(f) For a given conservative force field

$$\vec{F} = (5z\sin x + y)\hat{i} + (4yz + x)\hat{j} + (2y^2 - 5\cos x)\hat{k}$$

find the force potential. What is the work done on a particle starting at the origin and moving in a circular path of radius 2 to form a semicircle along the positive x-axis? 6

(Turn Over)

Or

A solid cylinder of mass 20 kg rotates about its own axis with angular velocity of 100 rad/s, the radius of the cylinder is 0·25 m. Calculate the kinetic energy associated with the rotation of the cylinder.

Paper : DSE–2.3

(Number Theory)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

1. *(a)* Write the Goldbach conjecture. 1

   (b) Can the Diophantine equation $14x+35y=93$ be solved? Give reasons to your answer. 2

   (c) Write the value of $\pi(30)$, where $\pi(x)$ denotes the prime counting function. 1

   (d) If $a \equiv b \pmod{m}$ and $x \equiv y \pmod{m}$, then prove that $ax \equiv by \pmod{m}$. 2

   (e) Find the remainder when 15! is divided by 17. 2

2. Answer any *three* of the following : 4×3=12

   (a) Find the general solution of the equation $5x+3y=52$.

   (b) Use Fermat's theorem to verify that 17 divides $11^{104}+1$.

   (c) Solve the simultaneous congruence :

   $$x \equiv 5 \pmod{7}$$
   $$x \equiv 7 \pmod{11}$$
   $$x \equiv 3 \pmod{13}$$

   (d) If p be a prime number, then show that $(p-1)! \equiv (p-1) \pmod{(1+2+3+\cdots+p-1)}$

   (e) Prove that if p is a prime, then $a^p \equiv a \pmod{p}$ for any integer a.

3. *(a)* Write the value of $\sigma(p)$, where p is prime. 1

   (b) Prove that for each positive integer n, $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3)=0$. 2

4. Answer any *four* of the following : 3×4=12

   (a) Prove that for any integer $n>1$

   $$n^{\frac{\tau(n)}{2}} = \prod_{d|n} d$$

(b) If F is a multiplicative function and

$$F(n) = \sum_{d|n} f(d)$$

then prove that f is also multiplicative.

(c) Prove that the function σ is multiplicative.

(d) Find the highest power of 7 that divides 2000!

(e) For $n > 2$, prove that $\phi(n)$ is even integer.

5. (a) Use Euler's theorem to establish any *one* of the following : 5

(i) For any integer $n \geq 0$, 51 divides $10^{32n+9} - 7$

(ii) For any integer a, $a^{37} \equiv a (\text{mod } 1729)$, given that $1729 = 7 \times 13 \times 19$

(b) For any positive integer n, prove that

$$\phi(n) = n \sum_{d|n} \frac{\mu(d)}{d}$$

Also verify it for $n = 12$. 5

(c) Define Dirichlet product of arithmetic functions. Also prove that $(f * g) * h = f * (g * h)$, where f, g and h are arithmetic functions and '$*$' denotes their Dirichlet product. 5

(Continued)

Or

Prove that if f and g are both multiplicative functions, then their Dirichlet product $f * g$ is also multiplicative.

6. (a) Find the order of 5 modulo 12 and hence determine the order of 5^6 modulo 12. 2

(b) Prove that if a has order $2k$ modulo an odd prime p, then $a^k \equiv -1 (\text{mod } p)$. 3

7. Answer any *five* of the following : 5×5=25

(a) If a is a primitive root of m, then show that a^k is also a primitive root of m if and only if $(k, \phi(m)) = 1$.

(b) Show that the primitive root of 13 are given by $S = \{2^n, 1 \leq n < \phi(m)\}$, $(n, \phi(m)) = 1$ when 2 is a primitive root of 13. Also find the exact number of primitive roots of 13.

(c) If $\gcd(m, n) = 1$, where $m > 2$ and $n > 2$, then prove that the integer mn has no primitive roots.

(d) Write the Euler's criterion for quadratic residue of an odd prime. Find which of the integers $1, 2, 3, \cdots, 12$ are quadratic residues of 13 and which are non residues of 13.

(Turn Over)

(e) State quadratic reciprocity law. Also find the value of $\left(\frac{29}{53}\right)$.

(f) Solve the following quadratic congruence :

$$x^2 + 7x + 10 \equiv 0 \pmod{11}$$

(g) Define Legendre symbol $\left(\frac{a}{p}\right)$, where p is an odd prime and $\gcd(a, p) = 1$. Prove that $a \equiv b \pmod{p} \Rightarrow \left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$. Hence show that $\left(\frac{3}{11}\right) = \left(\frac{14}{11}\right)$.

(h) Show that if p is an odd prime, then

$$\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) = 0$$

(i) Show that 7 and 18 are the only incongruent solution of $x^2 \equiv -1 \pmod{5^2}$.

(j) The message IWGA IU ZWU has been encoded with a Caesar cipher. Decipher it, using exhaustive cryptoanalysis.

Paper : DSE–2.4

(Biomathematics)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

UNIT—I

1. Answer any *two* of the following questions : 7½×2=15

(a) A population is originally 100 individuals, but because of the combined effects of births and deaths, it triples each hour.

(i) Make a table of population size for $t = 0$ to 5, where t is measured in hours.

(ii) Give two equations modelling the population growth by first expressing P_{t+1} in terms of P_t and then expressing ΔP in terms of P_t.

(iii) What can you say about the birthrate and death rate for this population?

(b) In the early stages of the development of a frog embryo, cell division occurs at a fairly regular rate. Suppose you observe that all cells divide, and hence the number of cells double, roughly every half-hour.

(i) Write down an equation modelling this situation. You should specify how much real-world time is required by an increment of 1 in t and what the initial number of cells is.

(ii) Produce a table and graph of the number of cells as a function of t.

(c) Obtain a simple prey-predator model explaining in detail the assumptions taken. Also find the equilibrium positions.

UNIT—II

2. Answer any *two* of the following questions : $7\frac{1}{2}\times 2=15$

(a) Consider the SI epidemic model. If the contact rate is 0·001 and the number of susceptible is 2000 initially, determine—

(i) the number of susceptible left after 3 weeks;

(ii) the density of susceptible when the rate of appearance of new cases is a maximum;

(Continued)

(iii) the time (in weeks) at which the rate of appearance of new cases is a maximum;

(iv) the maximum rate of appearance of new cases.

(b) In an SIS model, if the infection is spread only by a constant number of carriers, then show that

$$I(t)=\left(I_0-\frac{\alpha CN}{\alpha C+\beta}\right)e^{[-(\alpha C+\beta)t]}+\frac{\alpha CN}{\alpha C+\beta}$$

where I and C are the number of infectives and carriers; N total population; α and β are contact rate and susceptible rate respectively; I_0 is the infectives at $t=0$.

(c) Let x and y respectively denote the proportion of susceptibles and carriers in a population. Suppose the carriers are identified and removed from the population at a rate β, so that $\frac{dy}{dt}=\beta y$. Suppose also that the disease spreads at a rate proportional to the product of x and y, thus

$$\frac{dx}{dt}=-\alpha xy$$

(Turn Over)

(i) Determine the proportions of carriers at any time t, where $y(0) = y_0$.

(ii) Use (i) to find the susceptibles at time t, where $x(0) = x_0$.

(iii) Find the proportion of population that escapes the epidemic.

UNIT—III

3. Answer any *two* of the following questions : 7½×2=15

(a) Consider the competition model for two species with populations N_1 and N_2 :

$$\frac{dN_1}{dt} = r_1 N_1\left(1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_1}\right)$$

$$\frac{dN_2}{dt} = r_2 N_2\left(1 - b_{21}\frac{N_1}{K_2}\right)$$

where only one species N_1, has limited carrying capacity. Investigate their stability and sketch the phase plane trajectories. [Here, K_1 and K_2 are carrying capacities; r_1 and r_2 are linear birthrates of the populations N_1 and N_2 respectively. b_{12} and b_{21} measure the competitive effect of N_2 on N_1 and N_1 on N_2 respectively.] 4+3½=7½

(b) What is Routh-Hurwitz criteria? Explain with reference to multiple species communities. 2+5½=7½

(c) Discuss bifurcation and limit cycle with respect to any biological model.

UNIT—IV

4. Answer any *two* of the following questions : 7½×2=15

(a) Write a short note on any *one* of the following :

(i) One species model with diffusion

(ii) Two species model with diffusion

(b) For a blood vessel of constant radius R, length L and driving force $P = p_1 - p_2$, show that the average velocity of the flow is equal to half of the maximum velocity and the resistance is proportional to $\frac{L}{R^4}$.

(c) Consider the arterial blood viscosity $\mu = 0 \cdot 027$ poise. If the length of the artery is 2 cm, and radius 8×10^{-3} cm and $P = p_1 - p_2 = 4 \times 10^3$ dynes/cm^2, then find—

(i) $q_z(r)$ and the maximum peak velocity of blood;

(ii) the shear stress at the wall.

Here q_z denotes velocity along z-axis, p_1 and p_2 denote pressure at two ends of the artery.

UNIT—V

5. Answer any *two* of the following questions : 10×2=20

(a) Let D and d, and W and w respectively denote allele for tall and dwarf, and round and wrinkled seeds of peas. Find the outcome of the product DdWw × ddWw using Puneet square or using probability. Also find the probability that the progeny of DdWw × ddWw is dwarf with round seeds. 6+4=10

(b) Explain, in detail the Hardy-Weinberg equilibrium, mentioning the assumptions considered for the equilibrium.

(c) Compare and contrast stage structure model with age structure model.

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