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5 SEM TDC MTMH (CBCS) C 12

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(November)

MATHEMATICS

(Core)

Paper : C–12

(Group Theory—II)

Full Marks : 80
Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. (a) State True or False : 1

Every cyclic group is abelian.

(b) Define characteristic subgroup. 2

(c) If ϕ be an automorphism of a group G, then show that $H = \{x \in G \mid \phi(x) = x\}$ is a subgroup of G. 3

(d) Show that if $O(\text{Aut } G) > 1$, then $O(G) > 2$. 3

(e) Let G be a group. Show that the mapping $\phi : G \to G$ such that $\phi(x) = x^{-1}$ $\forall x \in G$ is an automorphism if and only if G is abelian. 4

(Turn Over)

(f) Let T be an automorphism of G. Show that $O(Ta) = O(a)$ for $a \in G$. Deduce that $O(baa^{-1}) = O(a)$ for all $a, b \in G$. 5

2. Answer any *two* of the following : 6×2=12

(a) Determine Aut(G), where G is Klein's 4-group.

(b) Prove that the characteristic subgroup of G must be a normal subgroup of G. The converse need not be true.

(c) Let $I(G)$ be the set of all inner automorphisms of a group G. Then prove that

$$I(G) \approx \frac{G}{Z(G)}$$

3. (a) Express $U(165)$ as an external direct product of cyclic group of the form Z_n. 2

(b) Find the number of cyclic subgroups of order 10 in $Z_{10} \oplus Z_{25}$. 3

(c) If m and n are relatively prime, then prove that $U(mn) \approx U(m) \oplus U(n)$. 5

Or

Find the external direct product of the following two cyclic groups :

$G_1 = \{a, a^2 = e_1\}$; $G_2 = \{b, b^2, b^3 = e_2\}$

(d) Prove that a group G is internal direct product of its subgroups H and K if and only if (i) H and K are normal subgroups of G and (ii) $H \cap K = \{e\}$. 5

(e) Let A and B be cyclic groups of orders m and n respectively. Prove that $A \times B$ is cyclic if and only if m and n are relatively prime. 5

Or

Show that a group of order 4 is either cyclic or an internal direct product of two cyclic subgroups each of order 2.

4. (a) Write the class equation for a finite group G. 1

(b) If a be an element of a group G, then show that G is abelian, if and only if $Cl(a) = \{a\} \ \forall a \in G$. 3

(c) Let G be a finite group and $Z(G)$ be the centre of G. Then prove that 3

$$O(G) = O(Z(G)) + \sum_{a \notin Z(G)} \frac{O(G)}{O(N(a))}$$

(d) If G is a finite group, then prove that

$$O(G) = \sum \frac{O(G)}{O(N(a))}$$

where the sum is taken over one element of each conjugate class. 3

(e) Let G be a finite group and a be an element of G. Then prove that

$$|Cl(a)| = \frac{|G|}{|N(a)|}$$ 4

(f) Prove that no group of order 30 is simple. 5

Or

Prove that a group of order 45 is abelian.

(g) Define Sylow p-subgroup. Prove that if p be a prime and k be a positive integer such that p^k divides $|G|$, where G is a finite group, then G has at least one subgroup of order p^k. 6

Or

State and prove Cauchy's theorem.

(h) Find two elements in A_5, the alternating group of degree 5, which are conjugate in S_5 but not in A_5. 5

Or

Prove that the number of Sylow p-subgroups of group G is of the form $1+kp$, where $(1+kp) \mid |G|$ and k being a non-negative integer.

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