

4 SEM TDC MTMH (CBCS) C 10

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(June/July)

MATHEMATICS

(Core)

Paper : C-10

(Ring Theory and Linear Algebra—I)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

1. (a) Give an example of a ring without unity. 1
- (b) Define unit element in a ring. 1
- (c) If the unity and the zero element of a ring R are equal, show that $R = \{0\}$, where 0 is the zero element of R . 2

- (d) Give an example of a subring which is not an ideal. 2
- (e) If I is an ideal of a ring R with unity such that $1 \in I$, show that $I = R$. 2
- (f) Show that \mathbb{Z}_{12} is not an integral domain. 2
- (g) Show that every field is an integral domain. Give an example to show that every integral domain is not necessarily a field. 4+1=5

Or

Define characteristic of a ring. Prove that the characteristic of an integral domain is 0 or a prime. 1+4=5

- (h) Show that if A and B are two ideals of a ring R , then $A+B$ is an ideal of R containing both A and B , where

$$A+B = \{a+b \mid a \in A, b \in B\} \quad 5$$

Or

Show that in a Boolean ring R , every prime ideal $P \neq R$ is maximal. 5

2. (a) Define kernel of a ring homomorphism. 1

(b) If $f: R \rightarrow R'$ be a ring homomorphism, show that $f(-a) = -f(a)$. 2

(c) Let R be a commutative ring with $\text{char}(R) = 2$. Show that $\phi: R \rightarrow R$ defined by $\phi(x) = x^2$ is a ring homomorphism. 2

(d) Let

$$R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in \mathbb{Z} \right\} \text{ and } \phi: R \rightarrow \mathbb{Z}$$

defined by

$$\phi \left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} \right) = a - b$$

Find $\ker \phi$. 2

(e) Let $f: R \rightarrow R'$ be an onto homomorphism, where R is a ring with unity. Show that $f(1)$ is the unity of R' . 3

Or

Prove that a homomorphism $f: R \rightarrow R'$ is one-one if and only if $\ker f = \{0\}$. 3

- (f) Show that the relation of isomorphism in rings is an equivalence relation. 5

Or

Let A, B be two ideals of a ring R . Show that

$$\frac{A+B}{A} \cong \frac{B}{A \cap B} \quad 5$$

3. (a) Is \mathbb{R} a vector space over \mathbb{C} ? 1

- (b) Define zero subspace of a vector space. 1

- (c) For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ of \mathbb{R}^2 and $\alpha \in \mathbb{R}$, let $x+y = (x_1+y_1, x_2+y_2)$ and $\alpha x = \alpha(x_1, x_2) = (\alpha x_1, 0)$. Is \mathbb{R}^2 a vector space with respect to above operations? Justify your answer. $1+1=2$

- (d) Let V be a vector space of all 2×2 matrices over the field \mathbb{R} of real numbers. Show that the set S of all 2×2 singular matrices over \mathbb{R} is not a subspace of V . 2

- (e) Consider the vectors $v_1 = (1, 2, 3)$ and $v_2 = (2, 3, 1)$ in $\mathbb{R}^3(\mathbb{R})$. Find k so that $u = (1, k, 4)$ is a linear combination of v_1 and v_2 . 2
- (f) Show that the vectors $v_1 = (1, 1, 0)$, $v_2 = (1, 3, 2)$ and $v_3 = (4, 9, 5)$ are linearly dependent in $\mathbb{R}^3(\mathbb{R})$. 3
- (g) Prove that any basis of a finite-dimensional vector space is finite. 4

Or

Let W_1 and W_2 be two subspaces of a finite-dimensional vector space. Then show that

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) \quad 4$$

4. (a) Let T be a linear transformation from a vector space U to a vector space V over the field F . Prove that the range of T is a subspace of V . 3
- (b) Examine whether the following mappings are linear or not : $2+2=4$

(i) $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by

$$T(x, y, z) = (|x|, y + z)$$

(ii) $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$T(x, y) = (x + y, x)$$

(c) If $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by

$$T(x, y) = (x + y, x - y, y)$$

is a linear transformation, find the rank and nullity of T .

$$4+4=8$$

(d) Let T be a linear operator on \mathbb{R}^2 defined by $T(x_1, x_2) = (x_1, 0)$. Find the matrix of T with respect to the basis $\{v_1, v_2\}$, where $v_1 = (1, 1)$ and $v_2 = (2, -1)$.

5

(e) Let $T : V \rightarrow U$ be a linear transformation. Show that

$$\dim V = \text{rank } T + \text{nullity } T$$

5

Or

Prove that a linear transformation $T : V \rightarrow U$ is non-singular if and only if T carries each linearly independent subset of V onto a linear independent subset of U .

5

- (f) Define isomorphism of vector spaces.
Prove that the mapping

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d)$$

from $M_2(\mathbb{R})$ to \mathbb{R}^4 is an isomorphism. 5

Or

Prove that every n -dimensional vector space over a field F is isomorphic to F^n . 5
